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THE CASE FOR GENERAL MATHEMATICS[1]

By WILLIAM DAVID REEVE
University of Minnesota

I shall not attempt, in this paper, to discredit our traditional methods of teaching algebra in the first year of the high school, followed by plane geometry in the second year, intermediate algebra in the third year, and so on. I say this in spite of the fact that much of our traditional practice and the accompanying results might justify one in so doing. In short, I am not interested in a destructive type of criticism of past methods with a view to setting up new bits of content (or at least reorganized content) and technique of procedure. Certainly, I should not favor a method which would seem to be attempting to force any set program upon the teaching body. The best progress is not made in that way. With many teachers of mathematics, the traditional order of treatment, if not the traditional methods, will prevail. Moreover, this will be true even after much experience and available scientific data may make a trial of some form of reorganized content and methods seem wise and feasible.

There are teachers in this country, however, who not only do not want to be limited to certain set programs, but who covet the opportunity to learn better methods of doing things by actually trying out materials and then observing and measuring results. It is because of such an attitude that the content and methods of teaching of first year algebra have been improved in recent years. A corresponding improvement has been made in the content and methods of teaching plane geometry, and I, for one, am grateful for whatever progress we have made. Teachers pretty generally, have agreed with the improvements referred to above, but they have been rather slow in changing their

[1] Read at the meeting of the National Council of Teachers of Mathematics at Chicago, March 1, 1920.

courses to agree with them. We shall continue to make progress along this line, but we must not expect a large part of our teaching body to assimilate new material very fast. We must be patient and lift our standards and reorganize our courses in such a way as not to be misunderstood. Even such a body as the College Entrance Board may not be expected to be as ready to make as radical changes as certain individual teachers may see fit to suggest.

We have come to believe in scientific methods in education. Those of us who happen to be connected with experimental schools of one kind or another have come to feel more deeply than ever before the responsibility that rests upon our shoulders in making some worth while contribution to our fields. We are not bound to proceed by set aims and objectives alone. We start a method of teaching, measure results, and thus arrive at certain conclusions that finally seem valid and important. After all, may not one seriously ask "What do we mean by set objectives?" Or perhaps, " Do we always secure the best results by emphasizing set objectives? " We are continually reminded by one educator or another that we are not securing these objectives. They even accuse us of not having any objectives at all and in many cases this is no doubt true. For example, the other day a friend of mine described to me an experiment that a research student was carrying on in a certain college of agriculture. He was experimenting in the feeding of hogs. No final objectives had been set up, but by careful observations and measurements he finally obtained some very valuable results to be worked for in the feeding of hogs.

I understand that the Latin teachers, in their national survey, are more concerned with doing things and the measuring results than they are in setting up and defending any set objectives in the teaching of Latin. This, it seems to me, is wise, not because we cannot set up certain objectives, but because the methods of measuring results is more likely to keep one open minded. The only difficulty here lies in not having time and money enough to continue the experiment long enough.

The proof of the pudding is in the eating. The case for general mathematics, so far as I am concerned, rests largely on the fact that it gives results that are highly gratifying. This is

especially true if the work is continued from two to four years in the high school. The movement for a general course in mathematics is certainly not a new idea. John Perry, of England, saw the importance of recognizing the content and method of teaching mathematics many years ago. He put the case as follows: "Great fields of thought are now open which were unknown to the Alexandrian philosophers. If we begin our study as the Alexandrian philosophers did, with their simplest ideas in arithmetic and geometry, we shall get stale before we know much more than they did. If we begin assuming more complex things to be true (although I do not like to assume that in truth any idea is more complex than another) as we have done in arithmetic, as we ought to do in other parts of mathematics without becoming stale we may know of all the modren discoveries. We shall thus get the same intellectual training with more knowledge." He says further, "In these days all men ought to study natural science. Such a study is practicably impossible without a knowledge of higher mathematical methods than that of the mere housekeeper. It must be more than what is called 'knowledge,' it must be mental dexterity, and it must be kept in constant practice if it is not to become rusty, and if men are to remain unafraid of mathematics. As examples of the methods necessary even in the most elementary study of nature I may mention: the use of logarithms in computation; knowledge of and power to manipulate algebraic formulae; the use of squared paper; the methods of the calculus. Dexterity in all of these is easily learned by all young boys. In such practice their brain power develops quite rapidly and they learn with pleasure. I feel sure that such dexterity cannot hinder, and can only further the mathematical study of the exceptionally clever sudent."

"For an advanced study of natural phenomena we need the results of the best study of the greatest mathematicians. To me mathematics is a powerful weapon with which to unlock the mysteries of nature. If a man knows how to use the weapon, that is enough. Let him leave to others, the men who delight in that, the forging of a weapon, the complete study of it. If I can use the weapon, let my study be of another kind—I think of a higher kind—to study the secrets which even an unskilled use of the weapon will reveal to me."

"I have the belief that the study of physical science, and therefore the study of mathematics, by everybody, however poor or however rich, is of the utmost importance to our country, not merely for the knowledge it gives, but for producing the scientific habit of thought, giving to every unit of the population a power to think for itself, and so producing the greatest happiness and giving the greatest strength of all kinds to the nation."

Classroom experience shows that when certain things in mathematics are necessary to the child's progress they should be given him, if need be, without proof. He should be taught how to use them until they become a part of his machinery and later, when he is more mature, he can study further and develop the nature of the machinery he has been using. This is what we all do every day. We simply take on faith a number of things we cannot fully understand. We can thus lead the child into much higher and more powerful mathematics without any ultimate loss. It is well known in our study of mathematics that we are continually making large assumptions where failure to do so would be not only unwise but ridiculous.

Many of us have been spending a whole semester on material that ought to be covered in considerably less time. In this connection I want to suggest that great economy in time could be achieved if only we would determine, scientifically, how long it takes to teach a certain topic satisfactorily. And by satisfactorily we should mean "to teach the topic to the degree of perfection indicated by some standard agreed upon." For example, how long does it take to teach, say a normal group, or even a certain kind of selected group, to learn how to factor the difference of two squares so that nobody in the group would make more than one error out of fifteen possible cases?

We cannot expect complete learning of a topic the first time over, as some writers would have us believe. Such a belief is contrary to the facts gained by class room experience. We should adopt some good spiral plan that will give good results.

Then too, what harm can come from omitting a great deal of the material that we have been teaching in algebra and geometry and taking for granted a great deal more that we have been forced to try to prove with little or no success? There is not a person today who cannot point out some phase of his math-

ematical training that has been of little real service compared to what might have been expected considering the time spent in acquiring that particular experience. The old fashioned geometrical method of proving the Pythagorean Theorem can be replaced by a simpler and shorter algebraic method known to every competent teacher. In fact, a great deal of Euclidean geometry can be proved by simple algebra if we permit unification of the two subjects.

The crowded condition of the freshman classes in some of our colleges and universities suggest further the advisability of furnishing more opportunities for a further pursuance of mathematics in the high school than is often the case. This may or may not lead eventually to a more serious consideration of the Junior College in many places. In any case, the plan, if followed, will enable the colleges to secure a better trained group of entering students and to devote more of their time to some of the advanced aspects of the subject. The trouble has been that we have spent so much time on the elementary phases of the subject trying to make sure that the child learns everything before he goes on, that we are not able to give him much in advance of what his ancestors had. As a result, a number of our colleges and universities are teaching little in the first two years beyond high school work. A great deal of the material we have to learn has never been of value to us either in mathematics itself or in the allied fields. Let us teach not so much mathematics, but more about mathematics.

My proposal is that wherever the conditions permit we arrange our mathematics so that we shall not have courses in algebra, in geometry, or in trigonometry as such, but a definitely arranged and psychologically ordered course in mathematics. The plan which we are now formulating will give the student who desires it a four year course in mathematics in the high school plus one or one and one half years of college work. It is also hoped that the course will be more compact, will involve less waste, develop more power, and produce even better results than we have been getting from traditional methods. The content of this course will be algebra, plane and solid geometry, trigonometry, most of the analytic geometry, and the fundamental elements of the calculus.

There is nothing essentially new in the plan of teaching algebra and geometry together. The best teaching talent in mathematics the world over, long ago recognized the importance of emphasizing the relation between algebra and geometry and the advisability of teaching them together. The movement to unify the two subjects has been vigorously opposed by those who seriously insist that such unification will break up the logically ordered system of Euclidean geometry and give rise to a series of unrelated ideas without unity or natural sequence. But if we are ever to cut the "ancestral process" short enough to enable us to give the boys and girls anything beyond what we ourselves have had, we must omit a great deal that was formerly taught and reorganize what is left in a more psychological way. We can still plan courses in algebra and geometry, as such, for the special needs and the special cases where such procedure seems necessary and wise.

The recent work and reports of the National Committee on Mathematical Requirements have furnished us with much valuable material and their final report will furnish us a basis for much further research work in the content and teaching technique concerning high school mathematics. But we must make sure that the good work of this committee is carried on.

The purpose of a general mathematics course in the high school should be to furnish a basis for a modern scholarly course in elementary mathematics that will give such careful training in power and appreciation as well informed citizens of our democracy ought to possess. And further, to arrange the material so that the boy or girl who is forced to leave the high school at the end of one or two years may nevertheless get a better understanding and appreciation of some of the finer things of life so that he may enjoy himself as he goes along. For the student that remains, it is believed a general course will save at least a year and that his ultimate conception of the entire field will be more thorough and fundamental.

In the traditional high school course, algebra is taught in the first year, geometry in the second, intermediate algebra and solid geometry or trigonometry in the third. It is very unusual that any mathematics is offered in the fourth year. Until recently these subjects were taught in water tight compartments so that

when the student studied algebra he felt that he was through with it, and so with plane geometry. He has not, as a rule, been taught to see how one subject may be made to reinforce and supplement the other. This artificial pigeon-holing of the subject matter has been a practice that good and well trained teachers of mathematics have never observed, but the rank and file have been very much hedged in by the traditional practice of treating the topics separately. Does it not seem rather a matter of tradition only that we have kept the algebra, the geometry, and the trigonometry in strictly parallel lines of treatment? Does this mean that we shall always sacrifice the psychological for what may often be considered the more logical order of treatment? "Teachers in lower grades have never realized that the union of logic and space studies deprived them of one of their most natural subjects of instruction, namely, form study. The logical statement of the principles of geometry has blinded modern as well as medieval teachers to the true worth of this subject for younger pupils." There is no doubt that the traditional tandem treatment of algebra first, geometry second, and so on, gives rise to a great deal more waste than many teachers realize, and it does not show the intermingling of the subjects as it should. A general mathematics course in the high school will show us more clearly how each subject is reinforced and made clearer and more helpful by the other.

The organizing and unifying principle of the general mathematics course in my own school is the idea of the functional relation—the dependence of one quantity upon another. In the first year we make the function the background of the course and we make the simpler truths and constructions of geometry help to rationalize some of the more formal aspects of the algebra and later to furnish exercises for algebraic applications. This is done in various ways, but the function concept, either implicitly or explicitly, dominant throughout, helps to lend concreteness and coherence to the subject. Excessive formalism is greatly reduced and the emphasis is placed upon the function, the equation, the formula, and the graph. Enough time is saved to permit us to furnish more illustrations and applications of principles and to introduce new and more important material.

Instead of waiting until the second year and trying to crowd all of the difficulties of demonstrative geometry into that year, we introduce much intuitional geometry in the first year, followed in some cases, by simple demonstrative geometry only where it comes naturally and easily. In this way many of the relations are taught inductively by experiment and by measurement, to be followed later by more formal proofs. The children who have geometry in this way do not start the geometry work in the second year as if it were an entirely new subject, as has often been the case.

In the next place, our traditional methods have delayed the teaching of much that is interesting and valuable in the secondary field. In this respect many of the English and continental schools of Europe are far in advance of us. The elementary ideas of numerical trigonometry, in many respects very simple, have been omitted altogether from most secondary work. They are as easy for a freshman in high school as anything else, once he understands similar triangles.

In the second year we take up the more logical and rigorous methods of demonstrative geometry as the central theme, but we keep the algebra before the student constantly by means of algebraic applications. In addition, we take up many of the facts of solid geometry at the points where their analogy is most natural and easy.

The work in trigonometry is carried on by giving more advanced problems in the solution of right triangles by logarithms, some work in proving simple identities, and some introduction into the solution of oblique triangles, where the law of sines applies. A great deal of emphasis in the second year is given to the different methods of attack and habits of studying. The student is made familiar with inductive, deductive, analytic, synthetic, and indirect methods of proof. By the end of the year we expect our capable pupils to do a high type of work.

In the third year the geometry is not so prominent and comes in again in the role of a helper to the algebra and trigonometry, although solid geometry is finished in this year's work. The algebra and trigonometry are related within topics as near as possible; e. g., when we are treating algebraic equations in one

unknown we treat trigonometric equations in one unknown and try to make clear the likenesses and differences in the nature of the solutions. We have ample opportunity to do a great deal with the simpler elements of analytic geometry without any loss of time so far as we can see.

In the fourth year we hope to unify the college algebra and the analytic geometry with some of the elements of the calculus in a better way than we have been able to do it so far by giving the earlier courses more compactness and better treatment. It goes without saying that not every student would be expected to study mathematics through four or even through three years, but it is interesting to note that where such opportunity is offered there are always students eager to register for the various courses though they are not required. At present we have fifteen seniors out of a class of fifty taking their fourth year of high school mathematics.

Our methods of teaching mathematics also need to be improved. The knowledge we now possess of individual differences in ability should make the study of mathematics a kind of laboratory course, in which more effective work can be done because the material can be better fitted to the individual pupil's needs. Such an arrangement of material decreases the need for so many reviews because each subject is kept in more or less constant use. As a result there is a gain in mathematical power and less need of home study. In the first year it is even possible to get along with little or no homework.

In order to give an idea of what some of the pupils taught in general mathematics course think of the method, the writer is quoting below a theme written by a freshman in one of his classes. The theme was handed in to an English teacher as a regular daily exercise. The title was "Mathematics, My Favorite Study." It ran as follows: "I think the first year of math is one of the best studies I have ever had for the following reasons: first, it is interesting; second, it is all, so far, based on one general idea, the equation; third, if absent, the work is easily made up. I can work for hours and hours at a time doing my "math" homework, because I like the study and it comes easy for me. I never count mathematics as a study; I think more of it as a sport,

like gym, not because I "rough house", but the time passes so quickly and I learn so many new things. In this way it is interesting.

"At the first of the year we thought of the equation as an expression of balance, as scales. Then we used the equation for angles, verbal and motion problems, graphs and finally for parallel lines. This brings up my interest, seeing all the different ways of using the same thing, and seeing that the equation can be used for almost all problems.

"As to being absent, that is easy. You have your book and you can go right ahead and do the problems as they are fully explained. If you can't get it this way, go to the study class when you come back to school, where you can get all the help you want, direct from the teachers. Can't you see that mathematics is a wonderful study, and why I like the first year of it so well?"

It is for the class of students who love mathematics or show unusual ability in it that I should want especially, to see the opportunity given to go on with their work in such a general course. They are the ones who will be the future teachers or research students in the subject.

At present I am not prepared to say what the colleges might do to adapt their courses to the students who finish the kind of course outlined above. That is largely a problem for the colleges to solve; but there are some things that will add greatly to the chance of an early agreement on a course. In the first place, the high school teachers should get together and formulate more definitely than heretofore minimum courses for each year of the high school, setting up certain standards of attainment. Secondly, the college teachers should familiarize themselves with these courses and, if they are satisfactory, they should build their college courses upon the high school courses as far as possible. In this connection, I have never understood why the general mathematics courses in the colleges have not been more successful. Thirdly, we ought to have frequent visitations back and forth wherever possible to enable those on both sides to keep in mind just what is to be expected as a final outcome and how much is being done on each side toward making a proper contribution. Then, if we study our habits, and improve our technique, we shall get results that are worth while and mathematics

will maintain the dignity which it has so long held in the curriculum. There has been an enormous lot of time wasted by repetition and so-called reviews that do not deserve the name. And the fact that the college courses do not fit on properly to the high school courses has led to a still further loss of many a student's time. Then too, a great many colleges and universities fail to provide courses at the proper time for students who are ready and eager to go on and often loss of time and gaps in instruction appear. It will be argued, of course, that proper adjustments are impossible because of administrative difficulties, but it is not hard to conceive of a little better situation than we often find.

It may be of interest here to say that of all our University high school students who have had at least two years of work in general mathematics, and who have subsequently taken further courses in the University, 13.2% have received marks of A; 31.6% marks of B; 34.2% marks of C; 13.2 marks of D; and 7.8% marks of F. Only two students have failed and one of these was one of our honor students at graduation and the other stood at the bottom of the class.

Moreover, of these who did not have at least two years of work in general mathematics, i. e. those who had their mathematics before they came to us, and who have taken further mathematics in the University, no one received a mark of A; 11.7% received B; 25.6% received C; 39.5% received D; and 23.2% were given a mark of failure. Of course I should not claim that these results establish the claims for a general mathematics course in the high school, but they are interesting.